UE D - Dynamique

# *ÉTUDE D'UN RADAR :*
# **Expliquer le comportement dynamique d'un système réel à l'aide de sa mise en équation et de son interprétation**

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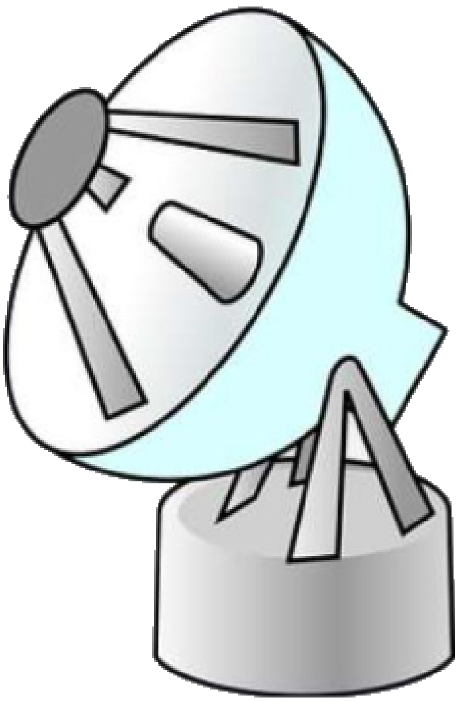

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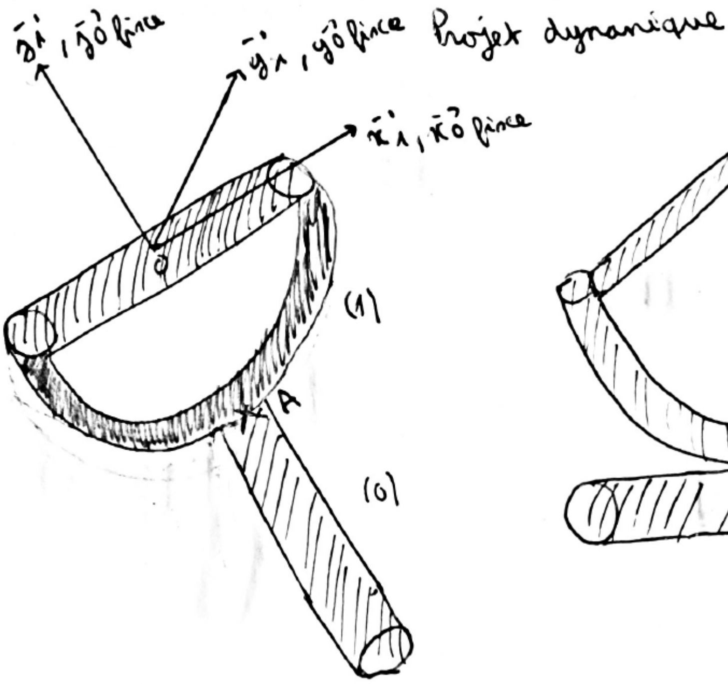
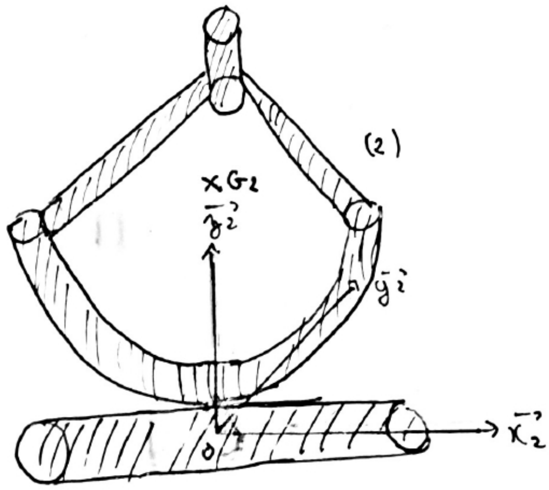
# Table des matières

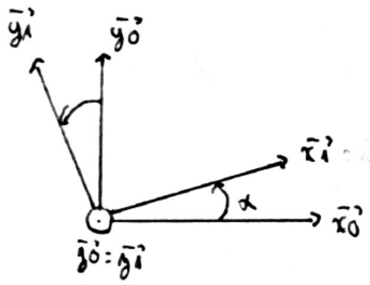


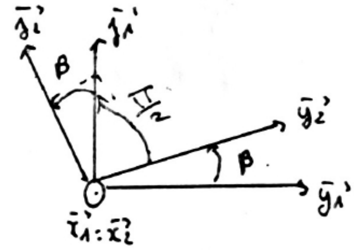
# Table des figures

# 1 Modélisation du système : solides, liaisons, repères, paramètres, points et les axes pour décrire les liaison




$\vec{x_1} = \cos\alpha \, \vec{x_0} + \sin(\alpha) \, \vec{y_0}$

$z_1 = \sin(\beta) \, \vec{y_2} + \cos(\beta) \, \vec{z_2}$

$\Omega_{1/0} = \dot{\alpha} \, \vec{z_0}$

$\Omega_{2/1} = \dot{\beta} \, \vec{x_1}$

$$\begin{aligned}\Omega_{2/0} &= \dot{\alpha} \, \vec{z_0} + \dot{\beta} \, \vec{x_1} \\ &= \dot{\beta} \, \vec{x_2} + \dot{\alpha} \left( \cos\left(\frac{\pi}{2} - \beta\right) \vec{y_2} + \sin\left(\frac{\pi}{2} - \beta\right) \vec{z_2} \right) \\ &= \dot{\beta} \, \vec{x_2} + \dot{\alpha} \sin(\beta) \, \vec{y_2} + \dot{\alpha} \cos(\beta) \, \vec{z_2}\end{aligned}$$

1

## 2 Schémas cinématiques (2D et 3D) + graphe des liaisons

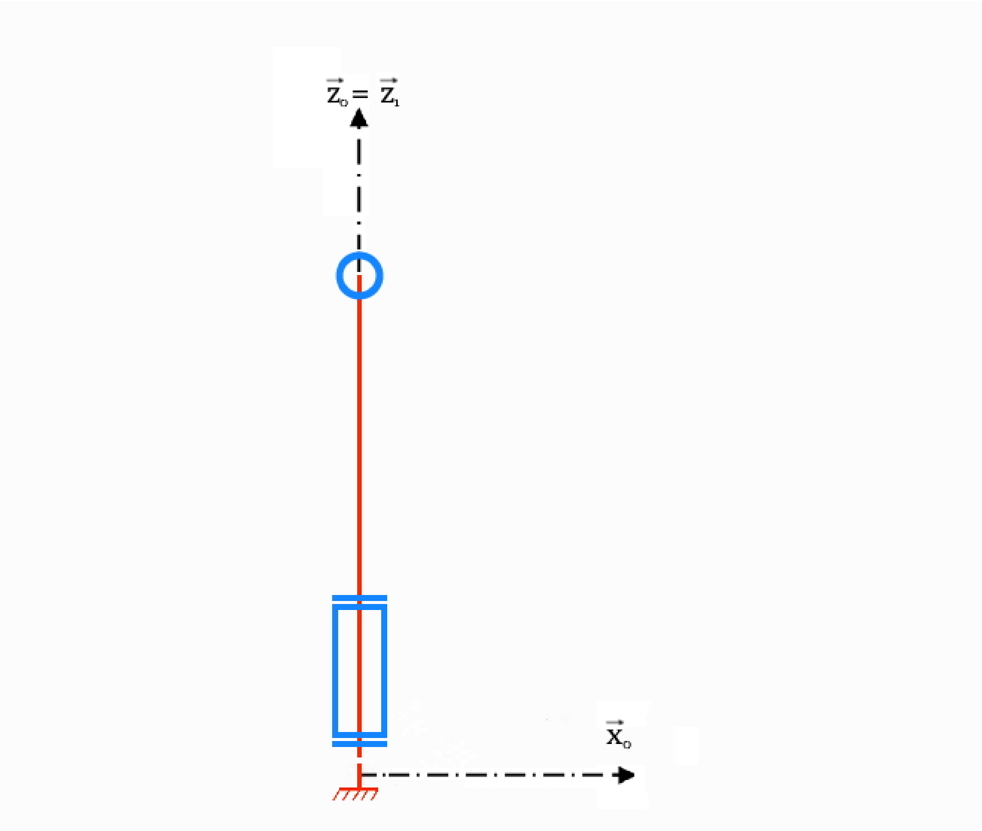



Figure 1 – Schéma cinématique 2D

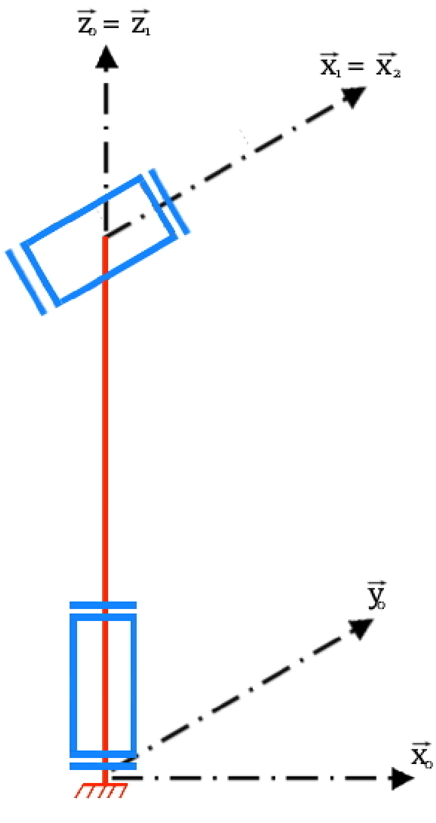



Figure 2 – Schéma cinématique 3D

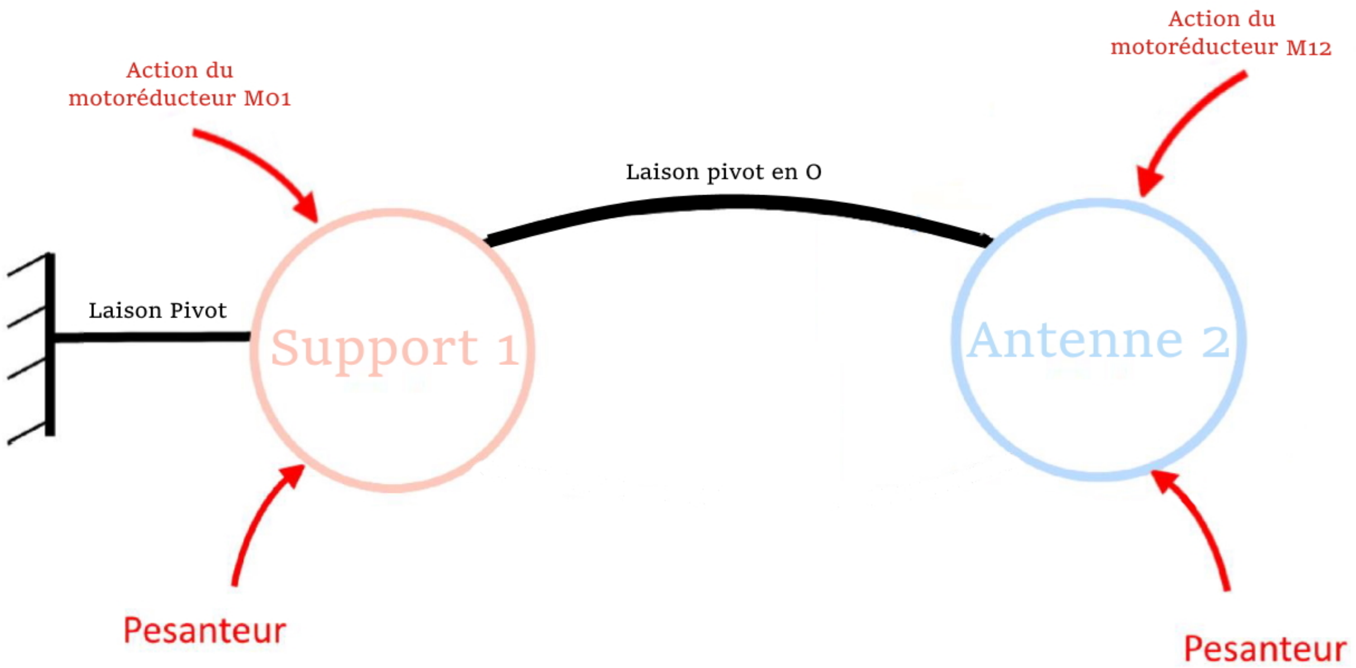
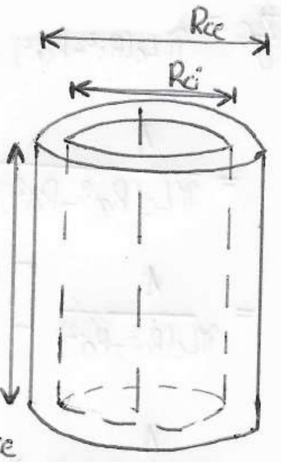



Figure 3 – Graphe des liaisons

# 3 Géométrie des masses

## 3.1 Centres de masse

### 3.1.1 Cylindre creux




On sait, d'ores et déjà, que pour calculer le centre de masse de notre solide l'expression de son volume sera une indispensable. C'est pourquoi l'on débute par le calcul de celui-ci.

<u>Calculer le volume d'un cylindre creux de rayons $R_{ce}$, $R_{ci}$ et de longueur $L_c$.</u>

$$V = \iiint_{M \in S} dV \quad \text{avec} \quad dV = r\,dr\,d\theta\,dz \qquad \begin{cases} R_{ci} \leq r \leq R_{ce} \\ 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq L_c \end{cases}$$

$$= \int_0^{L_c} \int_0^{2\pi} \int_{R_{ci}}^{R_{ce}} r\,dr\,d\theta\,dz$$

$$= \int_0^{L_c} dz \int_0^{2\pi} d\theta \int_{R_{ci}}^{R_{ce}} r\,dr$$

$$= \left[ z \right]_0^{L_c} \left[ \theta \right]_0^{2\pi} \left[ \frac{r^2}{2} \right]_{R_{ci}}^{R_{ce}}$$

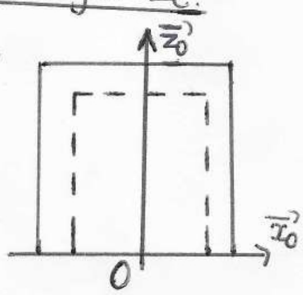
$$= L_c \times 2\pi \times \left( \frac{R_{ce}^2}{2} - \frac{R_{ci}^2}{2} \right)$$

$$\boxed{V = \pi L_c (R_{ce}^2 - R_{ci}^2)}$$

Maintenant que nous avons réalisé le calcul de notre volume, on peut aisément passer au calcul de son centre de masse.

<u>Calculer la position du cdm d'un cylindre creux de rayons $R_{ce}$, $R_{ci}$ et de longueur $L_c$.</u>

Par symétrie, on peut dire que : $\overrightarrow{OG} \begin{vmatrix} 0 \\ 0 \\ z_G \end{vmatrix}$. Soit M un point quelconque du solide : $\overrightarrow{OM} \begin{vmatrix} x \\ y \\ z \end{vmatrix}$.




$$m \cdot z_G = \int z \cdot dm, \text{ avec } dm = \rho \cdot dV, \ \rho = \frac{m}{V}, \ dV = r\,dr\,d\theta\,dz$$

Comme dit précédemment, on a $V = \pi L_c (R_{ce}^2 - R_{ci}^2)$

Donc $m \cdot z_G = \dfrac{m}{\pi L_c (R_{ce}^2 - R_{ci}^2)} \displaystyle\int z \cdot dV = \dfrac{m}{\pi L_c (R_{ce}^2 - R_{ci}^2)} \iiint z \cdot r \cdot dz\, d\theta.$

$$z_G = \frac{1}{\pi L_c(R_{ce}^2 - R_{ci}^2)} \int_0^{L_c}\int_0^{2\pi}\int_{R_{ci}}^{R_{ce}} z r dr\, dz d\theta$$

$$= \frac{1}{\pi L_c(R_{ce}^2 - R_{ci}^2)} \int_0^{L_c} z dz \int_0^{2\pi} d\theta \int_{R_{ci}}^{R_{ce}} r dr$$

$$= \frac{1}{\pi L_c(R_{ce}^2 - R_{ci}^2)} \left[\frac{z^2}{2}\right]_0^{L_c} \left[\theta\right]_0^{2\pi} \left[\frac{r^2}{2}\right]_{R_{ci}}^{R_{ce}}$$

$$= \frac{1}{\pi L_c(R_{ce}^2 - R_{ci}^2)} \times \frac{L_c^2}{\cancel{2}} \times \cancel{2}\pi \times \left(\frac{R_{ce}^2}{2} - \frac{R_{ci}^2}{2}\right)$$

$$= \frac{1}{\cancel{\pi} \cancel{L_c} \cancel{(R_{ce}^2 - R_{ci}^2)}} \times \cancel{\pi} L_c^{\cancel{2}} \times \frac{1}{2} \times \cancel{(R_{ce}^2 - R_{ci}^2)}$$

$$\boxed{z_G = \frac{L_c}{2}}$$

## Application numérique

$$z_G = \frac{L_c}{2} = \frac{600}{2} \boxed{= 300 mm}$$

### 3.1.2 Demi tore plein

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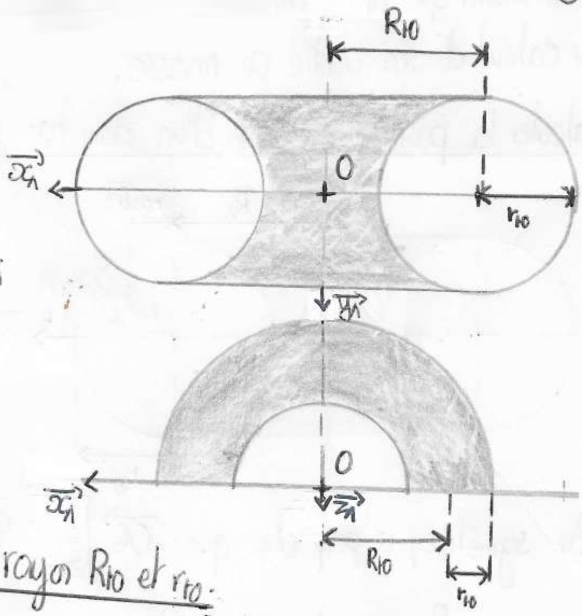
On sait, d'ores et déjà, que pour calculer le centre de masse de notre solide l'expression de son volume sera indispensable. C'est pourquoi l'on débute par le calcul de celui-ci.




Calculer le volume d'un demi tore plein de rayon $R_{10}$ et $r_{10}$.

$$V = \iiint_{MES} dV \quad \text{avec} \quad dV = (R_{10} + r\cos\theta)\, r\, dr\, d\varphi\, d\theta \qquad \begin{cases} 0 \leq r \leq r_{10} \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \end{cases}$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{r_{10}} (R_{10} + r\cos\theta)\, r\, dr\, d\varphi\, d\theta$$

$$= \int_0^{r_{10}}\int_0^{2\pi} (R_{10} + r\cos\theta)\, r\, dr\, d\theta \int_0^{\pi} d\varphi$$

$$= \pi \int_0^{2\pi} \left( \int_0^{r_{10}} R_{10} r + r^2 \cos\theta \, dr \right) d\theta$$

$$= \pi \int_0^{2\pi} \left[ R_{10}\frac{r^2}{2} + \cos\theta \frac{r^3}{3} \right]_0^{r_{10}} d\theta$$

$$= \pi \int_0^{2\pi} R_{10}\frac{r_{10}^2}{2} + \cos\theta \frac{r_{10}^3}{3} \, d\theta$$

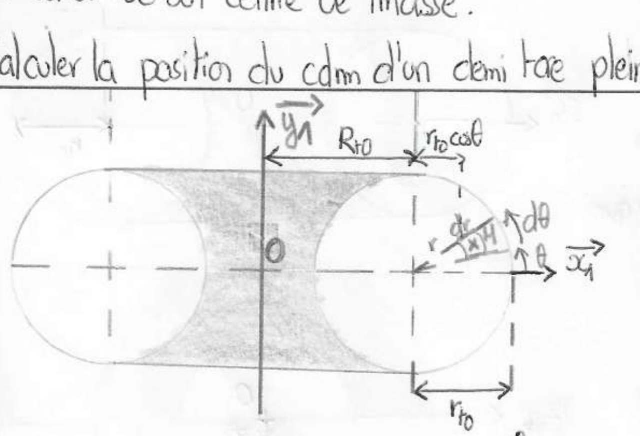
$$= \pi \left( R_{10}\frac{r_{10}^2}{2} \int_0^{2\pi} d\theta + \frac{r_{10}^3}{3} \int_0^{2\pi} \cos\theta \, d\theta \right)$$

$$= \pi \left( R_{10} r_{10}^2 \pi + \frac{r_{10}^3}{3} \left[ \sin\theta \right]_0^{2\pi} \right)$$

$$\boxed{= \pi^2 R_{10} r_{10}^2}$$

Maintenant que nous avons réalisé le calcul de notre volume, on peut aisément passer au calcul de son centre de masse.

Calculer la position du cdm d'un demi tore plein de rayons $R_{t0}$ et $r_{t0}$




$$\begin{cases} x_M = (R_{t0} + r\cos\theta)\cos\varphi \\ y_M = r\sin\theta \\ z_M = (R_{t0} + r\cos\theta)\sin\varphi \end{cases}$$

Par symétrie, on peut dire que : $\overrightarrow{OG}\Big|\begin{smallmatrix}0\\0\\z_G\end{smallmatrix}$. Soit M un point quelconque du solide : $\overrightarrow{OM}\Big|\begin{smallmatrix}x_M\\y_M\\z_M\end{smallmatrix}$.

$m \cdot z_G = \int z_M dm$, avec $dm = \rho \cdot dV$, $\rho = \frac{m}{V}$, $dV = (R_{t0} + r\cos\theta) r\, dr\, d\varphi\, d\theta$

Comme dit précédemment, on a $V = \pi^2 R_{t0} r_{t0}$

Donc $\cancel{m} \cdot z_G = \frac{\cancel{m}}{\pi^2 R_{t0} r_{t0}} \int z_M dV = \frac{m}{\pi^2 R_{t0} r_{t0}} \iiint z_M (R_{t0} + r\cos\theta) \cdot r \cdot dr \cdot d\varphi \cdot d\theta$

$$z_G = \frac{1}{\pi^2 R_{t0} r_{t0}^2} \iiint (R_{t0} + r\cos\theta)\sin\varphi\,(R_{t0} + r\cos\theta)\, r\, dr\, d\varphi\, d\theta$$

$$= \frac{1}{\pi^2 R_{t0} r_{t0}^2} \iiint (R_{t0} + r\cos\theta)^2 \sin\varphi\, r\, dr\, d\varphi\, d\theta$$

$$= \frac{1}{\pi^2 R_{t0} r_{t0}^2} \int_0^{2\pi} \left( \int_0^{r_{t0}} r(R_{t0} + r\cos\theta)^2 dr \right) d\theta \int_0^{\pi} \sin\varphi\, d\varphi$$

$$= \frac{1}{\pi^2 R_{t0} r_{t0}^2} \int_0^{2\pi} \left( \int_0^{r_{t0}} r(R_{t0} + r\cos\theta)^2 dr \right) d\theta \left( -(-1) + 1 \right)$$

$$= \frac{2}{\pi^2 R_{t0} r_{t0}^2} \int_0^{2\pi} \left( \int_0^{r_{t0}} R_{t0}^2 r + 2R_{t0}\cos\theta r^2 + r^3\cos^2\theta\, dr \right) d\theta$$

$$= \frac{2}{\pi^2 R_{t0} r_{t0}^2} \int_0^{2\pi} \left[ R_{t0}^2 \frac{r^2}{2} + 2R_{t0}\cos\theta \frac{r^3}{3} + \cos^2\theta \frac{r^4}{4} \right]_0^{r_{t0}} d\theta$$

$$= \frac{2}{\pi^2 R_{t0} r_{t0}^2} \int_0^{2\pi} R_{t0}^2 \frac{r_{t0}^2}{2} + 2R_{t0}\cos\theta \frac{r_{t0}^3}{3} + \cos^2\theta \frac{r_{t0}^4}{4}\, d\theta$$

## Centre de masse : Demi tore plein

(2)

$$z_G = \frac{2}{\pi^2 R_{to} r_{to}^2}\left(\frac{1}{2}R_{to}^2 r_{to}^2\int_0^{2\pi} d\theta + \frac{2}{3}R_{to} r_{to}^3\int_0^{2\pi}\cos\theta d\theta + \frac{1}{4}r_{to}^4\int_0^{2\pi}\cos^2\theta d\theta\right)$$

$$= \frac{2}{\pi^2 R_{to} r_{to}^2}\left(\pi R_{to}^2 r_{to}^2 + \cancel{\frac{2}{3}R_{to} r_{to}^3\left[\sin\theta\right]_0^{2\pi}} + \frac{1}{4}r_{to}^4\int_0^{2\pi}\cos^2\theta d\theta\right)$$

$$= \frac{2}{\pi^2 R_{to} r_{to}^2}\left[\pi R_{to}^2 r_{to}^2 + \frac{1}{8}r_{to}^4\left(\frac{1}{2}\int_0^{2\pi}\cos 2\theta d\theta + \int_0^{2\pi} d\theta\right)\right]$$

$$= \frac{2}{\pi^2 R_{to} r_{to}^2}\left[\pi R_{to}^2 r_{to}^2 + \frac{1}{8}r_{to}^4\left(\cancel{\frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{2\pi}} + 2\pi\right)\right]$$

$$= \frac{2}{\pi^2 R_{to} r_{to}^2}\left(\pi R_{to}^2 r_{to}^2 + \frac{1}{4}r_{to}^4\pi\right)$$

$$= \frac{2\pi R_{to}^2 \cancel{r_{to}^2}}{\pi^2 R_{to} \cancel{r_{to}^2}} + \frac{2 r_{to}^4 \pi}{4\pi^2 R_{to} r_{to}^2}$$

$$\boxed{z_G = \frac{2 R_{to}}{\pi} + \frac{r_{to}^2}{2\pi R_{to}}}$$

$$\cos^2\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2$$

$$= \frac{e^{i2\theta}+2e^{i\theta}e^{-i\theta}+e^{-i2\theta}}{4}$$

$$= \frac{e^{i2\theta}+e^{-i2\theta}}{4} + \frac{1}{2}$$

$$= \frac{\cos 2\theta}{4} + \frac{1}{2}$$

### Application numérique

$$z_G = \frac{2\times 250}{\pi} + \frac{5^2}{2\pi\times 250} \boxed{\simeq 159 mm}$$

### 3.1.3 Tige circulaire

→ Volume de la tige :

$$\iiint \rho\, d\rho\, d\theta\, dy$$

$$= \int_0^{R} \rho\, d\rho \int_0^{2\pi} d\theta \int_{-\frac{L_{Ti}}{2}}^{\frac{L_{Ti}}{2}} dy$$

$$= \left[\frac{1}{2}\rho^2\right]_0^{R} \cdot \left[\theta\right]_0^{2\pi} \cdot \left[y\right]_{-\frac{L_{Ti}}{2}}^{\frac{L_{Ti}}{2}}$$

$$= \frac{1}{2}R^2 \times 2\pi \times L_{Ti} = \pi R^2 L_{Ti}$$

Volume de la tige : $\pi R^2 L_{Ti}$

$$m\vec{Y_G} = \iiint \vec{Y_m}\, dm$$

avec $dm = \rho\, dV$, $\rho = \frac{m}{V}$, $\vec{Y_m} = \frac{L_{Ti}}{2}$

$$dV = \rho\, d\rho\, d\theta\, dy$$

$$\vec{Y_G} = \iiint \frac{1}{\pi R_{Ti}^2 L_{Ti}} \frac{L_{Ti}}{2} \rho\, d\rho\, d\theta\, dy$$

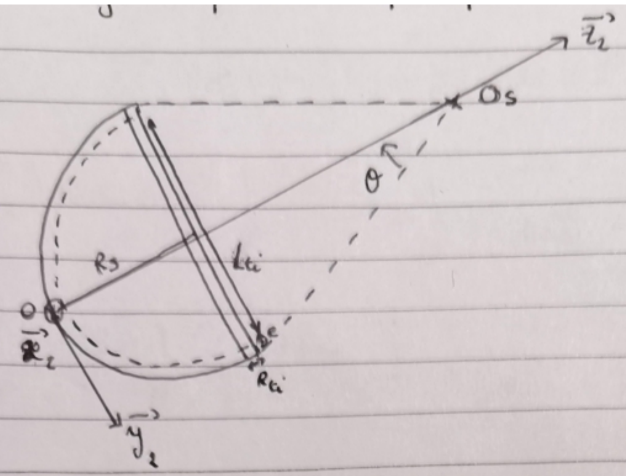
$$= \frac{1}{2\pi R_{Ti}^2} \int_0^{R_{Ti}} \rho\, d\rho \int_0^{2\pi} d\theta \int_{-\frac{L_{Ti}}{2}}^{\frac{L_{Ti}}{2}} dy$$

$$= \frac{1}{2\pi R_{Ti}^2} \left[\frac{1}{2}\rho^2\right]_0^{R_{Ti}} \left[\theta\right]_0^{2\pi} \left[y\right]_{-\frac{L_{Ti}}{2}}^{\frac{L_{Ti}}{2}}$$

$$= \frac{1}{2\pi R_{Ti}^2} \times \frac{1}{2} R_{Ti}^2 \times 2\pi \times L_{Ti}$$

$$\vec{Y_G} = \frac{L_{Ti}}{2} = \frac{500}{2} = 250 \text{ mm.}$$

### 3.1.4 Portion Sphérique




- $R_s = 300$ mm
  $\theta_s = 45°$
  $e_s = 4$ mm
- $R_{ti} = 5$ mm
  $L_{ti} = 500$ mm

* Centre de masse de la portion sphérique :

$$\begin{cases} R_s \leqslant r \leqslant R_s + e_s \\ 0 \leqslant \theta \leqslant \theta_s \\ 0 \leqslant \varphi \leqslant 2\pi \end{cases} \qquad \begin{cases} x_n = (r\sin\theta)\cos\varphi \\ y_n = (r\sin\theta)\sin\varphi \\ z_n = r\cos\theta \end{cases}$$

→ Volume de la portion sphérique :

$$\iiint r\,d\theta\; r\sin\theta\, d\varphi\, dr$$

$$= \iiint r^2 \sin\theta\, d\theta\, d\varphi\, dr$$

$$= \int_{R_s - e_s}^{R_s} r^2\, dr \int_0^{\theta_s} \sin\theta \int_0^{2\pi} d\varphi$$

$$= \left[\frac{1}{3} r^3\right]_{R_s - e_s}^{R_s} \times \left[-\cos\theta\right]_0^{\theta_s} \times \left[\varphi\right]_0^{2\pi}$$

$$= \frac{1}{3}\left(-R_s^3 + (R_s + e_s)^3\right) \times (1 - \cos\theta_s) \times 2\pi$$

$$= \frac{4\pi}{3}\left(-R_s^3 + (R_s + e_s)^3\right)(1 - \cos\theta_s)$$

$$\text{Volume } V = \frac{2\pi}{3}\left(-R_s^3 + (R_s + e_s)^3\right)(-1 - \cos\theta_s)$$

$m\vec{z_G} = \iiint \vec{z_m}\, dm$ avec $dm = \rho\, dV$ $\vec{z_m} = r\cos\theta$

$\rho = \frac{m}{V}$

$r \in [R_s ; R_s + e_s]$

$\theta \in [0 ; \frac{\pi}{4}]$ $dV = r^2 \sin\theta \; dr d\theta d\varphi$

$\varphi \in [0 ; 2\pi]$

$m\vec{z_G} = \iiint r\cos\theta \cdot \frac{3m}{2\pi(-R_s^3 + (R_s+e_s)^3)(1-\cos\theta_s)} r^2 \sin\theta\, dr d\theta dz$

$$\vec{z_G} = \frac{3\pi}{2\pi(R_s^3 - (R_s-e_s)^3)(1-\cos\theta_s)} \int_{R_s}^{R_s+e_s} r^3 \int_0^{\theta_s} \cos\theta\sin\theta\, d\theta \int_0^{2\pi} d\varphi$$

$$\vec{z_G} = \frac{3\pi}{2\pi(R_s^3 - (R_s-e_s)^3)(1-\cos\theta_s)} \left[\frac{1}{4} r^4\right]_{R_s}^{R_s+e_s} \left[\frac{\sin^2\theta}{2}\right]_0^{\theta_s} \left[\varphi\right]_0^{2\pi}$$

$$z_G = \frac{3\pi}{2\pi(R_s^3 - (R_s-e_s)^3)(1-\cos\theta_s)} \times \frac{1}{4}(-R_s^4 + (R_s+e_s)^4) \times \frac{1}{2}\sin^2\theta_s \times 2\pi$$

$$z_G = \frac{3}{(-0{,}3^3 + (0{,}3+4.10^{-3})^3)(1-\cos 45)} \times \frac{1}{8}(-R_s^4 + (R_s+e_s)^4) \times \sin^2\theta_s$$

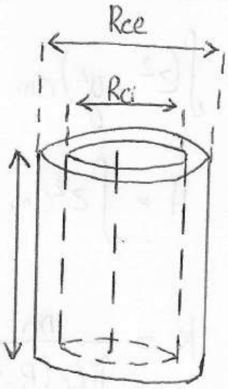
$z_G = 257{,}8$ mm.

## 3.2 Matrices d'inertie

### 3.2.1 Cylindre creux

Matrice d'inertie cylindre creux

$(O, \vec{z_0})$ est un axe de symétrie de révolution donc la matrice d'inertie du cylindre creux est diagonale et $A = B$




$$[I_{(O,\text{Cylindre creux})}] = \begin{Bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{Bmatrix}_{R_1}$$

$$\begin{cases} R_{ci} \leq r \leq R_{ce} \\ 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq L_c \end{cases}$$

Calcul de C

$$C = \iiint (x^2 + y^2)\,dm,\quad dm = \rho dV,\quad \rho = \frac{m}{V},\quad \overbrace{V = \pi L_c (R_{ce}^2 - R_{ci}^2)}^{\text{cf part 1. calc cdm}},\quad dV = r dr d\theta dz,\quad \overbrace{x^2 + y^2 = r^2}^{\text{car notre base est cylindrique}}$$

$$= \frac{m}{\pi L_c (R_{ce}^2 - R_{ci}^2)} \iiint r^3 dr d\theta dz$$

$$= \frac{m}{\pi L_c (R_{ce}^2 - R_{ci}^2)} \int_{R_{ci}}^{R_{ce}} r^3 dr \int_0^{2\pi} d\theta \int_0^{L_c} dz$$

$$= \frac{m}{\pi L_c (R_{ce}^2 - R_{ci}^2)} \left( \frac{R_{ce}^4 - R_{ci}^4}{4} \right) \times 2\pi \times L_c$$

$$= \frac{m (R_{ce}^4 - R_{ci}^4)}{2 (R_{ce}^2 - R_{ci}^2)}$$

$$\boxed{= \frac{m (R_{ce}^2 + R_{ci}^2)}{2}}$$

$$\boxed{\begin{gathered} \frac{a^4 - b^4}{a^2 - b^2} \\ \frac{(a^2 + b^2)\cancel{(a^2 - b^2)}}{\cancel{(a^2 - b^2)}} \\ a^2 + b^2 \end{gathered}}$$

## Calcul de A

$$A = \int (z^2 + y^2)\,dm \text{ , par symétrie } \int y^2\,dm = \int x^2\,dm = \frac{C}{2}$$

Donc $A = \int z^2\,dm + \frac{C}{2}$, on pose $K = \int z^2\,dm$

$$K = \frac{m}{\pi L_c (R_{ce}^2 - R_{ci}^2)} \iiint r\,dr\,d\theta\,z^2 dz$$

$$K = \frac{m}{\pi L_c (R_{ce}^2 - R_{ci}^2)} \int_{R_{ci}}^{R_{ce}} r\,dr \int_0^{2\pi} d\theta \int_0^{L_c} z^2 dz$$

$$K = \frac{m}{\cancel{\pi} L_c \cancel{(R_{ce}^2 - R_{ci}^2)}} \left( \frac{\cancel{R_{ce}^2 - R_{ci}^2}}{\cancel{2}} \right) \times \cancel{2\pi} \times \frac{L_c^3}{3}$$

$$\boxed{K = \frac{m L_c^2}{3}}$$

$$A \boxed{= \frac{m L_c^2}{3} + \frac{m(R_{ce}^2 + R_{ci}^2)}{4}}$$

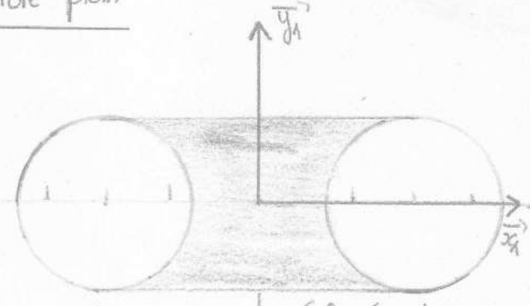
Conclusion

$$I_{(O,\text{ Cylindre creux})} = \begin{pmatrix} \frac{mL_c^2}{3} + \frac{m(R_{ce}^2+R_{ci}^2)}{4} & 0 & 0 \\ 0 & \frac{mL_c^2}{3} + \frac{m(R_{ce}^2+R_{ci}^2)}{4} & 0 \\ 0 & 0 & \frac{m(R_{ce}^2+R_{ci}^2)}{2} \end{pmatrix}_{R_1}$$

### 3.2.2 Demi tore plein

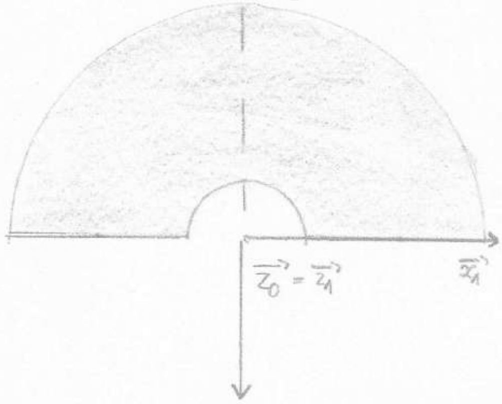
## Matrice d'inertie : Demi tore plein

Le demi-tore plein admet deux plans de symétrie orthogonaux ($Oyz$ et $Oxz$.) donc la matrice d'inertie de celui-ci est diagonale.




$$\begin{cases} 0 \leq r \leq r_{to} \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \end{cases}$$

$$\left[I\left(O, \tfrac{1}{2}\text{tore}\right)\right] = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}_{R_1}$$




$$A = \iiint (y^2+z^2)\,dm = \iiint y^2\,dm + \iiint z^2\,dm$$
$$B = \iiint (x^2+z^2)\,dm = \iiint x^2\,dm + \iiint z^2\,dm$$
$$C = \iiint (x^2+y^2)\,dm = \iiint x^2\,dm + \iiint y^2\,dm$$

On va donc poser :

$$X = \iiint x^2\,dm \qquad Y = \iiint y^2\,dm \qquad Z = \iiint z^2\,dm$$

$$dm = \rho\,dV \qquad \rho = \frac{m}{V} \qquad V = \pi^2 R_{to} r_{to}^2 \qquad dV = (R_{to} + r\cos\theta)\,r\,dr\,d\varphi\,d\theta$$

$$\begin{cases} x = (R_{to} + r\cos\theta)\cos\varphi \\ y = r\sin\theta \\ z = (R_{to} + r\cos\theta)\sin\varphi \end{cases}$$

puis chercher à déterminer la valeur de chacun d'entre eux.

$$X = \iiint x^2\,dm$$

$$X = \rho\iiint \left[(R_{to}+r\cos\theta)\cos\varphi\right]^2 (R_{to}+r\cos\theta)\,r\,dr\,d\varphi\,d\theta$$

Triangle de Pascal

1 1

1 2 1

1 3 3 1

$$= \rho\iiint r(R_{to}+r\cos\theta)^3\cos^2\varphi\,dr\,d\varphi\,d\theta$$

$$= \rho\int\int r\left(R_{to}^3 + 3R_{to}^2(r\cos\theta) + 3R_{to}(r\cos\theta)^2 + (r\cos\theta)^3\right)dr\,d\theta\int\cos^2\varphi\,d\varphi$$

$$= \rho\int_0^{2\pi}\int_0^{r_{to}} R_{to}^3 r + 3R_{to}^2\cos\theta\, r^2 + 3R_{to}\cos^2\theta\, r^3 + \cos\theta\, r^3\,dr\,d\theta \times \frac{1}{2}\int_0^{\pi}(\cos 2\varphi + 1)\,d\varphi$$

$$= \rho\int_0^{2\pi}\left(R_{to}^3\frac{r_{to}^2}{2} + 3R_{to}^2\cos\theta\frac{r_{to}^3}{3} + 3R_{to}\cos^2\theta\frac{r_{to}^4}{4} + \cos\theta\frac{r_{to}^4}{4}\right)\times\frac{1}{2}\times\left[\frac{\sin 2\varphi}{2}+\varphi\right]_0^{\pi}$$

$$= \rho\times\frac{\pi}{2}\left(\frac{R_{to}^3 r_{to}^2}{2}\int_0^{2\pi}d\theta + \frac{3R_{to}^2 r_{to}^3}{3}\int_0^{2\pi}\cos\theta\,d\theta + \frac{3R_{to}r_{to}^4}{4}\int_0^{2\pi}\cos^2\theta\,d\theta + \frac{r_{to}^4}{4}\int_0^{2\pi}\cos\theta\right.$$

$$= \rho\times\frac{\pi}{2}\left(\pi R_{to}^3 r_{to}^2 + \cancel{\frac{3R_{to}^2 r_{to}^3}{3}\left[\sin\theta\right]_0^{2\pi}} + \frac{3R_{to}r_{to}^4}{4}\times\frac{1}{2}\int_0^{2\pi}\cos\varphi + 1 + \cancel{\frac{r_{to}^4}{4}\left[\sin\theta\right]_0^{2\pi}}\right.$$

$$= \rho\times\frac{\pi}{2}\left(\pi R_{to}^3 r_{to}^2 + \frac{3R_{to}r_{to}^4}{8}\overbrace{\left[\sin\varphi+\varphi\right]_0^{2\pi}}^{2\pi}\right.$$

$$= \rho\times\frac{\pi}{2}\left(\pi R_{to}^3 r_{to}^2 + \frac{6\pi R_{to}r_{to}^4}{8}\right)$$

$$\boxed{X = \rho\left(\frac{\pi^2 R_{to}^3 r_{to}^2}{2} + \frac{3\pi R_{to} r_{to}^4}{8}\right)}$$

$$\boxed{\begin{aligned}\cos^2\varphi &= \left(\frac{e^{i\varphi}+e^{-i\varphi}}{2}\right)^2\\ &= \frac{e^{i2\varphi}+2e^{i\varphi}e^{-i\varphi}+e^{-i2\varphi}}{4}\\ &= \frac{e^{i2\varphi}+e^{-i2\varphi}}{4}+\frac{1}{2}\\ &= \frac{\cos(2\varphi)}{2}+\frac{1}{2}\\ &= \frac{1}{2}\left(\cos(2\varphi)+1\right)\end{aligned}}$$

$$Y=\rho\int_0^{2\pi}\int_0^{\pi}\int_0^{r_{t0}} r^3\sin^2\theta\,(R_{t0}+r\cos\theta)\,dr\,d\varphi\,d\theta$$

$$=\rho\int_0^{2\pi}\int_0^{r_{t0}} R_{t0}\sin^2\theta\, r^3 + \sin^2\theta\cos\theta\, r^4\;dr\,d\theta\int_0^{\pi}d\varphi$$

$$=\rho\pi\int_0^{2\pi}\left(R_{t0}\sin^2\theta\,\frac{r_{t0}^4}{4}+\sin^2\theta\cos\theta\,\frac{r_{t0}^5}{5}\right)d\theta$$

$$=\rho\pi\left(\frac{R_{t0}r_{t0}^4}{4}\int_0^{2\pi}\sin^2\theta + \frac{r_{t0}^5}{5}\int_0^{2\pi}\sin^2\theta\cos\theta\right)$$

$$=\rho\pi\left(\frac{R_{t0}r_{t0}^4}{8}\int_0^{2\pi}1-\cos(2\theta)\,d\theta+\frac{r_{t0}^5}{20}\int_0^{2\pi}\cos\theta-\cos3\theta\,d\theta\right)$$

$$=\rho\pi\left(\frac{R_{t0}r_{t0}^4}{8}\left[\theta-\frac{\sin2\theta}{2}\right]_0^{2\pi}+\frac{r_{t0}^5}{20}\left[\sin\theta-\frac{\sin3\theta}{3}\right]_0^{2\pi}\right)$$

$$=\rho\pi\left[\frac{R_{t0}r_{t0}^4}{8}\left(2\pi-\cancel{\frac{\sin(4\pi)}{2}}-\cancel{0}+\cancel{\frac{\sin 0}{2}}\right)+\frac{r_{t0}^5}{20}\left(\cancel{\sin 2\pi}-\cancel{\frac{\sin 6\pi}{3}}\right)\right]$$

$$=\rho\pi\left[\frac{R_{t0}r_{t0}^4}{8}(2\pi)\right]$$

$$\boxed{Y=\rho\frac{\pi^2R_{t0}r_{t0}^4}{4}}$$

$$\sin^2\theta=\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^2$$

$$=\frac{e^{i2\theta}-2e^{i\theta}e^{-i\theta}+e^{-i2\theta}}{-4}$$

$$=\frac{e^{i2\theta}+e^{-i2\theta}}{-4}+\frac{1}{2}$$

$$=-\frac{\cos(2\theta)}{2}+\frac{1}{2}$$

$$\boxed{\sin^2\theta=\frac{1}{2}\left(1-\cos(2\theta)\right)}$$

$$\sin^2\theta\cos\theta=\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^2\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)$$

$$=\frac{e^{i2\theta}-2e^{i\theta}e^{-i\theta}+e^{-i2\theta}}{-4}\times\frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$=\frac{e^{i3\theta}-2e^{i\theta}+e^{-i\theta}+e^{i\theta}-2e^{-i\theta}+e^{-i3\theta}}{-8}$$

$$=\frac{e^{i3\theta}+e^{-i3\theta}}{-8}+\frac{e^{i\theta}+e^{-i\theta}}{-8}+2\,\frac{e^{i\theta}-e^{-i\theta}}{8}$$

$$=-\frac{\cos(3\theta)}{4}-\frac{\cos(\theta)}{4}+2\,\frac{\cos(\theta)}{4}$$

$$\boxed{=\frac{1}{4}\left(\cos(\theta)-\cos(3\theta)\right)}$$

$$Z = \iiint z^2 \, dm$$

$$= \rho \iiint \left[ (R_{to} + r\cos\theta)\sin\varphi \right]^2 (R_{to} + r\cos\theta) r \, dr \, d\varphi \, d\theta$$

Triangle de Pascal

1 1

1 2 1

1 3 3 1

$$= \rho \iiint r (R_{to} + r\cos\theta)^3 \sin^2\varphi \, dr \, d\varphi \, d\theta$$

$$= \rho \int \int r \left( R_{to}^3 + 3R_{to}^2 (r\cos\theta) + 3R_{to}(r\cos\theta)^2 + (r\cos\theta)^3 \right) dr \, d\theta \int \sin^2\varphi \, d\varphi$$

$$= \rho \int_0^{2\pi} \int_0^{r_{to}} R_{to}^3 r + 3R_{to}^2 \cos\theta \, r^2 + 3R_{to}\cos^2\theta \, r^3 + \cos\theta \, r^3 \, dr \, d\theta \times \frac{1}{2} \int_0^{\pi} 1 - \cos 2\varphi \, d\varphi$$

$$= \rho \int_0^{2\pi} \left( R_{to}^3 \frac{r_{to}^2}{2} + 3R_{to}^2 \cos\theta \frac{r_{to}^3}{3} + 3R_{to}\cos^2\theta \frac{r_{to}^4}{4} + \cos\theta \frac{r_{to}^4}{4} \right) \times \frac{1}{2} \left[ \varphi - \frac{\sin 2\varphi}{2} \right]_0^{\pi}$$

$$= \rho \times \frac{\pi}{2} \left( \frac{R_{to}^3 r_{to}^2}{2} \int_0^{2\pi} d\theta + \frac{3R_{to}^2 r_{to}^3}{3} \int_0^{2\pi} \cos\theta \, d\theta + \frac{3R_{to} r_{to}^4}{4} \int_0^{2\pi} \cos^2\theta \, d\theta + \frac{r_{to}^4}{4} \int_0^{2\pi} \cos\theta \right.$$

$$\rho \times \frac{\pi}{2} \left( \pi R_{to}^3 r_{to}^2 + \cancel{\frac{3R_{to}^2 r_{to}^3}{3} \left[ \sin\theta \right]_0^{2\pi}} + \frac{3R_{to} r_{to}^4}{4} \times \frac{1}{2} \int_0^{2\pi} \cos\varphi + 1 + \cancel{\frac{r_{to}^4}{4} \left[ \sin\theta \right]_0^{2\pi}} \right.$$

$$\rho \times \frac{\pi}{2} \left( \pi R_{to}^3 r_{to}^2 + \frac{3R_{to} r_{to}^4}{8} \left[ \sin\varphi + \varphi \right]_0^{2\pi} \right.$$

$$\rho \times \frac{\pi}{2} \left( \pi R_{to}^3 r_{to}^2 + \frac{6\pi R_{to} r_{to}^4}{8} \right)$$

$$\boxed{Z = \rho \left( \frac{\pi^2 R_{to}^3 r_{to}^2}{2} + \frac{3\pi R_{to} r_{to}^4}{8} \right)}$$

$$\sin^2\varphi = \left( \frac{e^{i\varphi} - e^{-i\varphi}}{2i} \right)^2$$

$$= \frac{e^{i2\varphi} - 2e^{i\varphi}e^{-i\varphi} + e^{-i2\varphi}}{-4}$$

$$= -\frac{e^{i2\varphi} + e^{-i2\varphi}}{4} + \frac{1}{2}$$

$$= -\frac{\cos(2\varphi)}{2} + \frac{1}{2}$$

$$= \frac{1}{2}\left( 1 - \cos(2\varphi) \right)$$

On a obtenu :

$$X = \rho\left(\frac{\pi^2 R_{to}^3 r_{to}^2}{2} + \frac{3\pi R_{to} r_{to}^4}{8}\right)$$

$$Y = \rho\,\frac{\pi^2 R_{to} r_{to}^4}{4}$$

$$Z = X$$

On a donc

$$A = \rho\left(\frac{\pi^2 R_{to} r_{to}^4}{4} + \frac{\pi^2 R_{to}^3 r_{to}^2}{2} + \frac{3\pi R_{to} r_{to}^4}{8}\right)$$

$$= \frac{m}{\pi^2 R_{to} r_{to}^2}\left(\frac{\pi^2 R_{to} r_{to}^4}{4} + \frac{\pi^2 R_{to}^3 r_{to}^2}{2} + \frac{3\pi R_{to} r_{to}^4}{8}\right)$$

$$\boxed{= \frac{m\, r_{to}^2}{2} + \frac{m R_{to}^2}{2} + \frac{3m\, r_{to}^2}{8\pi}}$$

$$C \boxed{= A}$$

$$B = 2\rho\left(\frac{\pi^2 R_{to}^3 r_{to}^2}{2} + \frac{3\pi R_{to} r_{to}^4}{8}\right)$$

$$= \rho\left(\pi^2 R_{to}^3 r_{to}^2 + \frac{3\pi R_{to} r_{to}^4}{4}\right)$$

$$= \frac{m}{\pi^2 R_{to} r_{to}^2}\left(\pi^2 R_{to}^3 r_{to}^2 + \frac{3\pi R_{to} r_{to}^4}{4}\right)$$

$$\boxed{B = m R_{to} + \frac{3 r_{to}^2 m}{4\pi}}$$

## Matrice d'inertie : Demi tore plein

$$\left[ I\left(O, \tfrac{1}{2}\text{tore}\right) \right] = \begin{pmatrix} \frac{m(r_{to}^2 + R_{to}^2)}{2} + \frac{3mr_{to}^2}{8\pi} & 0 & 0 \\ 0 & mR_{to} + \frac{3r_{to}^2 m}{4\pi} & 0 \\ 0 & 0 & \frac{m(r_{to}^2+R_{to}^2)}{2} + \frac{3mr_{to}^2}{8\pi} \end{pmatrix}$$

## Matrice d'inertie : Support 1

$$I(O, S_1) = I\left(O, \tfrac{1}{2}\text{tore}\right) + I(O, \text{cyl. creux})$$

$$I(O,S_1) = \begin{bmatrix} \frac{m(r_{to}^2+R_{to}^2)}{2} + \frac{3mr_{to}^2}{8\pi} + \frac{mLc^2}{3} + \frac{m(R_{ce}^2+R_{ci}^2)}{4} & 0 & 0 \\ 0 & mR_{to} + \frac{3r_{to}^2 m}{4\pi} + \frac{mLc^2}{3} + \frac{m(R_{ce}^2+R_{ci}^2)}{4} & 0 \\ 0 & 0 & \frac{m(r_{to}^2+R_{to}^2)}{2} + \frac{3mr_{to}^2}{8\pi} + \frac{mLc^2}{3} + \frac{m(R_{ce}^2+R_{ci}^2)}{4} \end{bmatrix}$$

### 3.2.3 Tige circulaire

Calculs des matrices d'inertie :

→ Tige

Il y a un axe de symétrie de révolution $(O;\vec{y_2})$ donc la matrice est diagonale et $A=C$

$A = \iiint (y^2+z^2)\,dm$ avec $z=0$ car le solide est filaire

$\int y^2\,dm$ avec $dm = \rho\,dy$ et $\rho = \frac{m}{L_{Ti}}$

$= \int_{-\frac{L_{Ti}}{2}}^{\frac{L_{Ti}}{2}} y^2 \frac{m}{L_{Ti}}\,dy$

$= \frac{m}{L_{Ti}} \left[\frac{1}{3}y^3\right]_{-\frac{L_{Ti}}{2}}^{\frac{L_{Ti}}{2}}$

$= \frac{m}{L_{Ti}} \left[\frac{1}{6}L_{Ti}^3 + \frac{1}{6}L_{Ti}^3\right] = \frac{m}{L_{Ti}} \times \frac{1}{3}L_{Ti}^3 = \frac{1}{3}m L_{Ti}^2 = C$

$B = \int (x^2+z^2)\,dm$ avec $x=z=0$ car le solide est filaire ; $B=0$.

$$I_{(O,Tige)} = \begin{bmatrix} \frac{1}{3}mL_{Ti}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}mL_{Ti}^2 \end{bmatrix}_{R_2}$$

### 3.2.4 Portion Sphérique

→ Portion sphérique

Il y a deux axes de symétrie $(O, x_1, y_2)$ et $(O, z_1, y_2)$ qui sont orthogonaux

$$A = \iiint \rho\, dV (y^2+z^2) \text{ avec } \rho = \frac{m}{V} \text{ et } dm = \rho\, dV \; ; \; dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$A = \frac{m}{V} \int \left( r^2 \sin^2\theta \cos^2\varphi + r^2 \cos^2\theta \right) r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= \frac{m}{V} \int \left( r^4 \sin^3\theta \cos^2\varphi + r^4 \cos^2\theta \sin\theta \right) dr\, d\theta\, d\varphi$$

$$= \frac{m}{V} \int r^4 dr \int \sin^3\theta\, d\theta \int \cos^2\varphi\, d\varphi + \frac{m}{V} \int r^4 . \int \cos^2\theta \sin\theta\, d\theta . \int d\varphi$$

$$= \frac{m}{V} \left[ \frac{1}{5} r^5 \right]_{R_S}^{R_S - e_S} \cdot \left[ -\cos\theta + \frac{1}{3} \cos^3(\theta) \right]_0^{\frac{\pi}{4}} \left[ \frac{\varphi}{2} + \frac{\sin 2\varphi}{4} \right]_0^{2\pi}$$
$$+ \frac{m}{V} \left[ \frac{1}{5} r^5 \right]_{R_S}^{R_S + e_S} \cdot \left[ -\frac{1}{3} \cos^3\theta \right]_0^{\pi/4} \cdot \left[ \varphi \right]_0^{2\pi}$$

$$= \frac{m}{V} \left[ \frac{1}{5} \left( -R_S^5 + (R_S + e_S)^5 \right) \times \left[ \left( -\frac{1}{\sqrt{2}} + \frac{1}{3} \times \frac{1}{2\sqrt{2}} \right) - \left( -1 + \frac{1}{3} \right) \right] \times \pi \right]$$

$$+ \frac{m}{V} \left[ \frac{1}{5} \left( -R_S^5 + (R_S + e_S)^5 \right) \times \left( -\frac{1}{3} \times \frac{1}{2\sqrt{2}} - \left( -\frac{1}{3} \times 1 \right) \right) \times 2\pi \right]$$

$$= \frac{m}{V} \left[ \frac{1}{5} \left( -R_S^5 + (R_S + e_S)^5 \right) \times \left( \frac{-6}{6\sqrt{2}} + \frac{1}{6\sqrt{2}} + 1 - \frac{1}{3} \right) \times \pi \right]$$

$$+ \frac{m}{V} \left[ \frac{1}{5} \left( -R_S^5 + (R_S + e_S)^5 \right) \times \left( -\frac{1}{6\sqrt{2}} + \frac{1}{3} \right) \times 2\pi \right]$$

$$= \frac{m}{V} \left[ \frac{\pi}{5} \left( (R_S + e_S)^5 - R_S^5 \right) \times \left( \frac{-5}{6\sqrt{2}} + \frac{6\sqrt{2} - 2\sqrt{2}}{6\sqrt{2}} \right) \right] + \frac{m}{V} \left[ \frac{2\pi}{5} \left( (R_S + e_S)^5 - R_S^5 \right) \times \left( -\frac{1}{6\sqrt{2}} + \frac{2\sqrt{2}}{6\sqrt{2}} \right) \right]$$

$$= \frac{m}{V} \left[ \frac{\pi}{5} \left( (R_S + e_S)^5 - R_S^5 \right) \times \left( \frac{-5 + 4\sqrt{2}}{6\sqrt{2}} \right) \right] + \frac{m}{V} \left[ \frac{2\pi}{5} \left( (R_S + e_S)^5 - R_S^5 \right) \times \left( \frac{-1 + 2\sqrt{2}}{6\sqrt{2}} \right) \right]$$

$$= \frac{m}{V} \left[ \frac{\pi}{30\sqrt{2}} \times \left( (R_S + e_S)^5 - R_S^5 \right) \left( -5 + 4\sqrt{2} \right) \right] + \frac{m}{V} \left[ \frac{2\pi}{30\sqrt{2}} \left( (R_S + e_S)^5 - R_S^5 \right) \left( -1 + 2\sqrt{2} \right) \right]$$

$$= \frac{m}{V} \times \left( (R_S + e_S)^5 - R_S^5 \right) \left[ \frac{\pi(-5 + 4\sqrt{2}) + 2\pi(-1 + 2\sqrt{2})}{30\sqrt{2}} \right] = 0{,}4177 \text{ kg.m}^2$$

$$B = \int (x^2 + z^2)\, dm \quad \text{avec} \quad dm = \rho dV\ ,\ \rho = \frac{m}{V}\ ,\ dV = r^2 \sin\theta\, dr d\theta d\varphi.$$

$$x = (r\sin\theta)\cos\varphi \qquad z = r\cos\theta$$

$$B = \iiint \left( (r^2 \sin^2\theta)\cos^2\varphi + r^2\cos^2\theta \right) \rho r^2 \sin\theta\, dr d\theta d\varphi$$

$$= \rho \iiint \left( r^4 \sin^3\theta \cos^2\varphi + r^4 \cos^2\theta \sin\theta \right) dr d\theta d\varphi$$

$$= \rho \iiint r^4 \sin^3\theta \cos^2\varphi\, dr d\theta d\varphi + \rho \iiint r^4 \cos^2\theta \sin\theta\, dr d\theta d\varphi$$

$$= \rho \int_{R_s}^{R_s+e_s} r^4 dr \cdot \int_0^{\frac{\pi}{4}} \sin^3\theta d\theta \cdot \int_0^{2\pi} \cos^2\varphi d\varphi + \rho \int_{R_s}^{R_s+e_s} r^4 dr \cdot \int_0^{\pi/4} \cos^2\theta \sin\theta d\theta \cdot \int_0^{2\pi} d\varphi$$

$$= \rho \left[\frac{1}{5} r^5\right]_{R_s}^{R_s+e_s} \cdot \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi/4} \cdot \left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{4}\right]_0^{2\pi} + \rho \left[\frac{1}{5} r^5\right]_{R_s}^{R_s+e_s} \cdot \left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi/4} \cdot \left[\varphi\right]_0^{2\pi}$$

$$= A = 0{,}4177 \text{ kg.m}^2.$$

$$C = \int (x^2 + y^2)\, dm \quad \text{avec} \quad dm = \rho dV\ ,\ \rho = \frac{m}{V}\ ,\ dV = r^2 \sin\theta\, dr d\theta d\varphi$$

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi$$

$$C = \rho \iiint \left( r^2 \sin^2\theta \cos^2\varphi + r^2 \sin^2\theta \sin^2\varphi \right) r^2 \sin\theta\, dr d\theta d\varphi$$

$$= \rho \iiint \left( r^4 \sin^3\theta \cos^2\varphi + r^4 \sin^3\theta \sin^2\varphi \right) dr d\theta d\varphi$$

$$= \rho \int_{R_s}^{R_s+e_s} r^4 dr \cdot \int_0^{\pi/4} \sin^3\theta\, d\theta \cdot \int_0^{2\pi} \cos^2\varphi d\varphi + \rho \int_{R_s}^{R_s+e_s} r^4 dr \cdot \int_0^{\pi/4} \sin^3\theta d\theta \cdot \int_0^{2\pi} \sin^2\varphi d\varphi.$$

$$= \rho \left[\frac{1}{5} r^5\right]_{R_s}^{R_s+e_s} \cdot \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi/4} \cdot \left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{4}\right]_0^{2\pi} + \rho \left[\frac{1}{5} r^5\right]_{R_s}^{R_s+e_s} \cdot \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi/4} \cdot \left[\frac{1}{2}(\varphi - \sin\varphi\cos\varphi)\right]_0^{2\pi}$$

$$= \rho \left[\frac{1}{5}\left((R_s+e_s)^5 - R_s^5\right) \times \left( \left(-\frac{1}{\sqrt{2}} + \frac{1}{3} \times \frac{1}{2\sqrt{2}}\right) - \left(-1 + \frac{1}{3}\right) \right) \times \pi \right] + \rho \left[ \left((R_s+e_s)^5 - R_s^5\right) \times \left( \left(-\frac{1}{\sqrt{2}} + \frac{1}{3} \times \frac{1}{2\sqrt{2}}\right) - \left(-1 + \frac{1}{3}\right) \right) \times \pi \right]$$

$$= 2\rho \left[ \frac{1}{5}\left((R_s+e_s)^5 - R_s^5\right) \left( \frac{-6+1}{6\sqrt{2}} + 1 - \frac{1}{3} \right) \times \pi \right]$$

$$= 2\pi\rho \left[ \frac{1}{5}\left((R_s+e_s)^5 - R_s^5\right) \left( \frac{-6 + 1 + 6\sqrt{2} - 2\sqrt{2}}{6\sqrt{2}} \right) \right]$$

$$= 2\pi\rho \left[ \frac{\left((R_s+e_s)^5 - R_s^5\right)\left(-5 + 4\sqrt{2}\right)}{30\sqrt{2}} \right] = 0{,}1272 \text{ kg.m}^2$$

$$I_{(0,05)} = \begin{bmatrix} 0,4177 & 0 & 0 \\ 0 & 0,4177 & 0 \\ 0 & 0 & 0,1272 \end{bmatrix}$$

# 4 Les 2 équations du PFD (équations de mouvement)

## 4.1 Equation du moment dynamique en projection sur l'axe (D21)

BAME : On isole 2

$R_0\{O, \vec{x_0}, \vec{y_0}, \vec{z_0}\}$, $R_1\{O, \vec{x_1}, \vec{y_1}, \vec{z_1}\}$, $R_2\{O, \vec{x_2}, \vec{y_2}, \vec{z_2}\}$

- action de la liaison pivot en O

$$_O\begin{Bmatrix} X_1 & 0 \\ Y_1 & M_1 \\ Z_1 & N_1 \end{Bmatrix}_{(x_2, y_2, z_2)}$$

- action du moteur en O

$$_O\begin{Bmatrix} 0 & Cm_{x2} \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_{(x_2, y_2, z_2)}$$

- action de pesanteur

$$_{G_2}\begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ -mg & 0 \end{Bmatrix}_{(-, -, z_0)}$$

$$M_O = M_{G_2} + \vec{OG_2} \wedge -mg\vec{z_0}$$
$$= 0 + Z_2\vec{z_2} \wedge -mg\vec{z_0}$$
$$= Z_2\, mg\, \vec{x_2} \sin(\beta)$$

$$_O\begin{Bmatrix} 0 & Z_2 mg \sin(\beta) \\ 0 & 0 \\ -mg & 0 \end{Bmatrix}_{(x_2, -, z_2)}$$

BQA :

* moment dynamique au point O

$$\vec{N}(O, 2/R_0) = \frac{d^{(0)} L(O, 2/R_0)}{dt}$$

$$\vec{L}(O, 2/R_0) = I(O,2) \times \Omega_{2/0} = \begin{bmatrix} A_2 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & C_2 \end{bmatrix}_{R_2} \times \begin{bmatrix} \dot{\beta} \\ \dot{\alpha}\sin(\beta) \\ \dot{\alpha}\cos(\beta) \end{bmatrix}_{R_2}$$

$$L(O, 2/R_0) = \begin{bmatrix} \dot{\beta} A_2 \\ B_2 \dot{\alpha}\sin(\beta) \\ C_2 \dot{\alpha}\cos(\beta) \end{bmatrix}_{R_2}$$

2

$$\frac{d^{(0)} L(O, 2/R_0)}{dt} \cdot \vec{x_2} = \frac{d^{(0)} \quad \dot{B} A_2}{dt}$$

$$= \begin{bmatrix} \dot{\beta} A_2 \\ B_2 \sin(\beta) \dot{\alpha} \\ C_2 \cos(\beta) \dot{\alpha} \end{bmatrix} \cdot \underbrace{\frac{d^{(0)} \vec{x_2}}{dt}}$$

$$\begin{cases} = \dfrac{d^{(1)} \vec{x_2}}{dt} + \dot{\alpha} \vec{z_1} \wedge \vec{x_1} = \dfrac{d^{(2)} \vec{x_2}}{dt} + \dot{\alpha} \vec{y_2} + \dot{\beta} \vec{x_1} \wedge \vec{x_2} \\ = \dot{\alpha} \vec{y_2} \end{cases}$$

$$= \boxed{\ddot{B} A_2 - B_2 \sin(\beta)(\dot{\alpha})^2}$$

Le système 2 est en mouvement par rapport au repère Galiléen on peut donc appliquer le PFD.

$$\vec{M}(O, ext/2) \cdot \vec{x_2} = N(O, 2/R_0) \cdot \vec{x_2}$$

$$\Rightarrow \boxed{\ddot{\beta} A_2 - B_2 \sin(\beta)(\dot{\alpha})^2 - m g z_2 \sin(\beta) = C_{m12}}$$

### 4.2 Equation du moment dynamique en projection sur l'axe (D10)

BAME : On isole $S\{1,2\}$

$R_0\{O, \vec{x_0}, \vec{y_0}, \vec{z_0}\}$, $R_1\{O, \vec{x_1}, \vec{y_1}, \vec{z_1}\}$, $R_2\{O, \vec{x_2}, \vec{y_2}, \vec{z_2}\}$

- action de la liaison pivot au point O

$$_O\begin{Bmatrix} X_2 & L_2 \\ Y_2 & M_2 \\ Z_2 & 0 \end{Bmatrix}_{(-,-,z_1)}$$

- action du moteur en O

$$_O\begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & C_{mot} \end{Bmatrix}_{(-,-,z_1)}$$

$\vec{M_O} = \vec{M_{G_S}} + \vec{OG_S} \wedge -mg\,\vec{z_0}$

- action de pesanteur en O

$= 0 + z_s\,\vec{z_1} \wedge -mg\,z_0 = 0$

$$_O\begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ -mg & 0 \end{Bmatrix}_{(-,-,z_0)} \qquad _O\begin{Bmatrix} 0 & z_2\,mg\sin(\beta) \\ 0 & 0 \\ -mg & 0 \end{Bmatrix}_{R_1}$$

le solide S est en mouvement par rapport au repères galiléens on peut donc appliquer le PFD.

BQA

* moment dynamique au point O

$\vec{N}(O, S/R_0) = \vec{N}(O,1/R_0) + \vec{N}(O,2/R_0)$

Le point O est fixe par rapport a $R_0$ : $N(O,1/R_0) = \dfrac{d^{(0)}\,\vec{L}(O,1/R_0)}{dt}$

3

$$\vec{L}(0,1/R_0) = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & C_1 \end{bmatrix}_{R_1} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix}_{R_1} = \begin{bmatrix} 0 \\ 0 \\ C_1 \dot{\alpha} \end{bmatrix}_{R_1}$$

$$N(0,1/R_0) \cdot \vec{z_1} = \frac{d^{(0)} C_1 \dot{\alpha}}{dt} = C_1 \ddot{\alpha}$$

$$N(0,2/R_0) \cdot \vec{z_1} = \frac{d^{(0)} \; B_2 \dot{\alpha} \sin(\beta)^2 + C_2 \dot{\alpha} \cos(\beta)^2}{dt}$$ dérivée pas terme $\sin(\beta)$

$$- \begin{bmatrix} \dot{\beta} A_2 \\ B_2 \sin(\beta) \dot{\alpha} \\ C_2 \cos(\beta) \dot{\alpha} \end{bmatrix} \cdot \underbrace{\frac{d^{(0)} \vec{z_1}}{dt}}$$

$$\frac{d^{(1)} \vec{z_1}}{dt} \; \dot{\alpha} z_1 \wedge \vec{z_1} = 0$$

$$N(0,2/R_0) \cdot \vec{z_1} = B_2 \left( \ddot{\alpha} \sin(\beta)^2 + 2 \dot{\beta} \dot{\alpha} \cos(\beta) \right) + C_2 \left( \ddot{\alpha} \cos(\beta)^2 - 2 \dot{\beta} \dot{\alpha} \sin(\beta) \right)$$

$$\boxed{C_{mo1} = B_2 \left( \ddot{\alpha} \sin(\beta)^2 + 2 \dot{\beta} \dot{\alpha} \cos(\beta) \right) + C_2 \left( \ddot{\alpha} \cos(\beta)^2 - 2 \dot{\beta} \dot{\alpha} \sin(\beta) \right) + C_1 \ddot{\alpha}}$$

$$I(O, \text{Tige}) = \begin{bmatrix} \frac{1}{3} m L_{Ti}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} m L_{Ti}^2 \end{bmatrix}_{R_2}$$

$$I(O, PS) = \begin{bmatrix} 0{,}4177 & 0 & 0 \\ 0 & 0{,}4177 & 0 \\ 0 & 0 & 0{,}1272 \end{bmatrix}_{R_2}$$

$$I(O, \text{cylindre creux}) = \begin{bmatrix} \frac{m L_c^2}{3} + \frac{m(R_{ce}^2 + R_{ci}^2)}{4} & 0 & 0 \\ 0 & \frac{m L_c^2}{3} + \frac{m(R_{ce}^2 + R_{ci}^2)}{4} & 0 \\ 0 & 0 & \frac{m(R_{ce}^2 + R_{ci}^2)}{2} \end{bmatrix}_{R_1}$$

$$I(O, \tfrac{1}{2}\text{tore}) = \begin{bmatrix} \frac{m(r_{to}^2 + R_{to}^2)}{2} + \frac{3 m r_{to}^2}{8\pi} & 0 & 0 \\ 0 & m R_{to} + \frac{3 r_{to}^2 m}{4\pi} & 0 \\ 0 & 0 & \frac{m(r_{to}^2 + R_{to}^2)}{2} + \frac{3 m r_{to}^2}{8\pi} \end{bmatrix}_{R_1}$$

4.